

ON A CONJECTURE OF CANDELAS AND DE LA OSSA

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ABSTRACT. We prove that the metric completion of a canonical Ricci-flat Kähler metric on the nonsingular part of a projective Calabi-Yau variety X with ordinary double point singularities, is a compact metric length space homeomorphic to the projective variety X itself. As an application, we prove a conjecture of Candelas and de la Ossa for conifold flops and transitions.

1. Introduction

Yau's solution to the Calabi conjecture in [49] gives the existence of a unique Ricci-flat Kähler metric in any given Kähler class. Calabi-Yau manifolds and Ricci-flat Kähler metrics play a central role in the study of string theory. A natural problem in both mathematics and physics is to understand how Calabi-Yau manifolds of distinct topological types can be connected via algebraic, analytic and geometric processes. In the algebraic aspect, this is exactly the well-known Reid's fantasy [31] built on deep works of Clemens [11], Friedman [14], Hirzebruch [23] and many others (cf. [44, 5, 6, 45, 19, 20, 21, 33]). A geometric transition is an algebraic notion of connectedness for the moduli space of Calabi-Yau threefolds, which involves with a birational contraction and a complex smooth deformation. It can be considered as the three dimensional analogue of analytic deformations among $K3$ surfaces. A conifold transition is a special geometric transition, where the contracted variety has only ordinary double points as singularity. The first physical interpretation of a conifold transition is given by Strominger [43]. In geometric and analytic aspect, the geometric transition should be considered for Calabi-Yau varieties coupled with canonical metrics such as Ricci-flat Kähler metrics. In [5], Candelas and de la Ossa conjecture that an algebraic conifold transition should be also analytic and geometric, i.e., the transition should be continuous in suitable geometric sense (Conjecture 2.1). The recent work of Rong and Zhang [32] proves a version of their conjecture by showing that an algebraic geometric transition is indeed continuous in Gromov-Hausdorff topology. The goal of the paper is to establish a strong version of Candelas and de la Ossa's conjecture for conifold transitions. In general, a geometric transition is not necessarily projective or even Kähler [14, 45]. The balanced metrics on non-Kähler Calabi-Yau threefolds are proposed in the direction to study non-Kähler geometric transitions [15].

We now state the main results of the paper. Let $f : X \rightarrow Y$ be a small contraction morphism of a smooth projective Calabi-Yau threefold X such that Y is a normal Calabi-Yau variety with only ordinary double points as singularities. Let \mathcal{L}_0 be an

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ample line bundle over Y and α be a Kähler class on X . Then there also exists a unique Ricci-flat Kähler metric $g(t) \in c_1(\alpha + t[\pi^*\mathcal{L}_0])$ for $t \in (0, 1]$ by Yau's theorem. There exists a unique singular Ricci-flat Kähler metric g_Y associated to its Kähler current $\omega_Y \in c_1(\mathcal{L}_0)$ obtained in [13]. In particular, ω_Y has bounded local potentials on Y and g_Y is a smooth Kähler metric on Y_{reg} , the nonsingular part of Y [13].

Theorem 1.1. *The metric completion of (Y_{reg}, g_Y) is a compact length space homeomorphic to the projective variety Y itself, denoted by (Y, d_Y) . Furthermore, $(X, g(t))$ converges to (Y, d_Y) in Gromov-Hausdorff topology, as $t \rightarrow 0$.*

Theorem 1.1 shows that the algebraic small contraction can be realized by a continuous deformation of smooth Calabi-Yau Kähler metrics in Gromov-Hausdorff topology. In fact, much stronger estimates are obtained in section 4 for degeneration of the Calabi-Yau metrics near the exceptional rational curves of f . Theorem 1.1 can be also applied to conifold flops and transitions as stated in the following corollaries. We also remark that the convergence is in fact smooth outside the exceptional rational curves as shown in [46].

Corollary 1.1. *Let*

$$(1.1) \quad \begin{array}{ccc} X & \cdots \cdots \cdots & X' \\ & \searrow f & \swarrow f' \\ & Y & \end{array}$$

be a conifold flop between two smooth projective Calabi-Yau threefolds X and X' . Let (Y, d_Y) be the compact metric length space induced by the singular Ricci-flat Kähler metric g_Y as in Theorem 1.1. Then there exist a smooth family of smooth Ricci-flat Kähler metrics $g(t)$ of X and a smooth family of smooth Ricci-flat Kähler metrics $g'(s)$ of X' for $t, s \in (0, 1]$, such that $(X, g(t))$ and $(X', g'(s))$ converge to (Y, d_Y) in Gromov-Hausdorff topology as $t, s \rightarrow 0$.

It shows that any discrete conifold flop between two Calabi-Yau threefolds can be connected by a continuous path of Calabi-Yau metrics in Gromov-Hausdorff topology.

Theorem 1.1 can also be applied to conifold transitions of Calabi-Yau threefolds. An algebraic geometric transition (cf. [33, 32]) is a triple $T(X, Y, Y_s)$ connecting Calabi-Yau threefolds of different topological types, where Y is a singular Calabi-Yau variety obtained from X by a birational contraction morphism and Y_s is a smooth complex deformation of Y . A conifold transition is a geometric transition such that the contracted singular Calabi-Yau variety Y has only ordinary double points as singularities. The precise definitions are given in Section 2. The following corollary shows that an algebraic conifold transition is also a diffeo-geometric transition via continuous families of Ricci-flat Kähler metrics.

Corollary 1.2. *Let $T(X, Y, Y_s)$ be a conifold transition of projective Calabi-Yau threefolds*

$$(1.2) \quad X \xrightarrow{f} Y \longleftarrow Y_s$$

for $s \in \Delta$, where Δ is the unit disc in \mathbb{C} . Then there exist a smooth family of smooth Ricci-flat Kähler metrics $g(t)$ of X for $t \in (0, 1]$ and a smooth family of smooth Ricci-flat Kähler metrics g_{Y_s} of Y_s for $s \in \Delta^*$, such that $(X, g(t))$ and (Y_s, g_{Y_s}) converge to (Y, d_Y) in Gromov-Hausdorff topology as $t, s \rightarrow 0$. Here (Y, d_Y) is the compact length metric space given in Theorem 1.1.

Corollary 1.2 proves a conjecture of Candelas and de la Ossa (Conjecture 2.1) for conifold transitions by combining Theorem 1.1 and the results of Rong and Zhang [32] (Theorem 2.2). If there exists an algebraic conifold transition between Calabi-Yau threefolds of distinct topology, it can be also constructed as a continuous transition for algebraic Calabi-Yau varieties coupled with canonical Ricci-flat Kähler metrics in the Gromov-Hausdorff "moduli space". The convergence in Corollary 1.1 and Corollary 1.2 is stronger than Gromov-Hausdorff convergence and in fact, it is in local C^∞ -topology outside the exceptional locus as shown in [46].

The organization of this article is as follows: In section 2, we give the background of conifold transitions, singular Ricci-flat Kähler metrics and complex Monge-Ampère equations. In section 3, we review the Calabi symmetry and construct various local ansatz near an exceptional rational curve. In section 4, we obtain various uniform estimates for a degenerate family of Ricci-flat Kähler metrics. Finally, we prove the main results in section 5 with some discussion on generalizations to higher dimensional conifold transitions and to canonical surgery by the Kähler-Ricci flow.

2. Conifold transitions and complex Monge-Ampère equations

In this section, we give a brief introduction on conifold transitions and canonical Ricci-flat Kähler metrics on singular Calabi-Yau varieties..

Definition 2.1. *Let X and X' be two smooth Calabi-Yau threefolds with $f : X \rightarrow Y$ and $f' : X' \rightarrow Y$ being small contraction morphisms. Then the following diagram is called a flop between X and X'*

$$(2.3) \quad \begin{array}{ccc} X & \cdots \cdots \cdots \rightarrow & X' \\ & \searrow f & \swarrow f' \\ & Y & \end{array}$$

Y in (2.3) is a normal variety with canonical singularities and trivial canonical divisor. There exists a unique Ricci-flat Kähler metric in any given polarization by the work of Essydieux, Guedj and Zeriahi [13]. This is shown by solving degenerate complex Monge-Ampère equations related to constant scalar curvature metrics (cf. [13, 52, 26, 29, 30, 28]). Let \mathcal{L}_0 be an ample line bundle over Y and so it induces an embedding morphism $Y \hookrightarrow \mathbb{P}^N$ into some big projective space. Let α be a Kähler class on X and we define $\alpha_t = \alpha + t[f^*\mathcal{L}_0]$. Obviously α_t is a Kähler class on X whenever $t > 0$. Let $\theta \in [\mathcal{L}_0]$ be a multiple of the pullback of the Fubini-Study metric on \mathbb{P}^N , $\omega_0 \in [\alpha]$ a Kähler metric on X and $\omega_t = \theta + t\omega_0$. Let Ω_{CY} be a smooth volume form on X such that $\sqrt{-1}\partial\bar{\partial}\log\Omega_{CY} = 0$. Then the solution of the following

complex Monge-Ampère equation

$$(2.4) \quad (\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^3 = c(t)\Omega_{CY}, \quad \sup_X \varphi_t = 0$$

gives rise to a Ricci flat Kähler metric $g(t)$ associated to the Kähler form

$$(2.5) \quad \omega(t) = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t,$$

where $c(t)$ is a family of constants in t determined by $\alpha_t^3 = c(t) \int_X \Omega_{CY}$.

The following deep estimates are obtained in [13, 52] built on techniques of Kolodziej [25].

Theorem 2.1. *There exists $C > 0$ such that for all $t \in (0, 1]$,*

$$(2.6) \quad \|\varphi_t\|_{L^\infty(X)} \leq C.$$

The local C^∞ regularity follows from the L^∞ estimate, Tsuji's trick [48] and the general linear theory (cf. [13, 36, 46]).

Proposition 2.1. *Let D be the exceptional locus of $f : X \rightarrow Y$ and $X^\circ = X \setminus D$. For any compact set K of X° and $k \geq 0$, there exists $C_{K,k} > 0$ such that for all $t \in (0, 1]$,*

$$(2.7) \quad \|\varphi_t\|_{C^k(K)} \leq C_{K,k}.$$

From the above uniform estimates, we obtain a unique solution for the limiting degenerate complex Monge-Ampère equation [13, 46]. It is shown in [46, 51] that the diameter of $(X, g(t))$ is uniformly bounded for $t \in (0, 1]$. The following corollary follows from Theorem 2.1 and Proposition 2.1 by letting $t \rightarrow 0$.

Corollary 2.1. *Let Y_{reg} be the nonsingular part of Y . There exists a unique $\varphi_0 \in PSH(Y, \theta) \cap L^\infty(Y) \cap C^\infty(Y_{reg})$ such that $\sup_X \varphi_0 = 0$ and*

$$(2.8) \quad (\theta + \sqrt{-1}\partial\bar{\partial}\varphi_0)^3 = c_\theta\Omega_{CY}, \quad \int_X \theta^3 = c_\theta \int_X \Omega_{CY}.$$

The Kähler current

$$(2.9) \quad \omega_Y = \theta + \sqrt{-1}\partial\bar{\partial}\varphi_0$$

induces the unique singular Ricci-flat Kähler metric on Y in $c_1(\mathcal{L}_0)$.

By the general theory in Riemannian geometry [7, 8, 9, 10], one can always take the Gromov-Hausdorff limit for the family of $(X, g(t))$ with $t \in (0, 1]$. On the other hand, $g(t)$ converges to g_Y in $C^\infty(X^\circ)$. Naturally, one would ask if the intrinsic limit of $(X, g(t))$ in Gromov-Hausdorff topology is homeomorphic to Y as a projective variety, and if it coincides with its extrinsic limit.

We now introduce the notion of geometric transitions for Calabi-Yau threefolds (cf. [33, 34, 32]).

Definition 2.2. *Let X be a Calabi-Yau threefold and $f : X \rightarrow Y$ be a contraction morphism from X to a normal Calabi-Yau variety Y . Suppose that Y admits a smooth projective deformation $\pi : \mathcal{M} \rightarrow \Delta$ over the unit disc $\Delta \in \mathbb{C}$ such that $K_{\mathcal{M}/\Delta} = \mathcal{O}_{\mathcal{M}}$ with smooth fibres $Y_s = \pi^{-1}(s)$ of Calabi-Yau three folds for $s \neq 0$ and $Y = Y_0$.*

Then the following diagram is called a geometric transition $T(X, Y, Y_s)$

$$(2.10) \quad X \xrightarrow{f} Y \longleftarrow Y_s .$$

Definition 2.3. A geometric transition $T(X, Y, Y_s)$ is called a conifold transition if Y admits only ordinary double points as singularity.

Definition 2.4. A flop between two Calabi-Yau threefolds X and X' as in (2.3) is called a conifold flop if Y admits only ordinary double points as singularity.

A local model for conifold singularities is given by

$$\{z \in \mathbb{C}^4 \mid z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}.$$

A well-known example for a conifold transition is given in [18]. Let Y be the hypersurface in \mathbb{P}^4 defined by

$$z_3 g(z_0, \dots, z_4) + z_4 h(z_0, \dots, z_4) = 0$$

with generic homogeneous polynomials g, h of degree 4 in $[z_0, z_1, \dots, z_4] \in \mathbb{P}^4$. The singular locus of Y is given by $\{z_3 = z_4 = g(z) = h(z) = 0\}$, which consists of 16 ordinary double points. The small resolution of the singularities of Y gives rise to a smooth Calabi-Yau threefold X and Y can also be smoothed to generic smooth quintic threefolds in \mathbb{P}^4 .

Let $T(X, Y, Y_s)$ be a geometric transition associated to a smoothing $\mathcal{M} \rightarrow \Delta$. Let \mathcal{L} be an ample line bundle over \mathcal{M} and $\mathcal{L}_s = \mathcal{L}|_{Y_s}$. Then there exists a unique smooth Calabi-Yau Kähler metric $g_{Y_s} \in c_1(\mathcal{L}_s)$ for $s \in \Delta^*$. When $s = 0$, there exists a unique singular Ricci-flat Kähler metric g_Y by Theorem 2.1 such that the associated Kähler current $\omega_Y \in c_1(\mathcal{L}_0)$ has bounded local potential and ω_Y^3 is a Calabi-Yau volume form on Y . In fact, g_Y is smooth on Y_{reg} , the nonsingular part of Y .

Let α be a Kähler class of X . Then $\alpha_t = \alpha + t[\mathcal{L}_0]$ is a Kähler class of X for $t \in (0, 1]$ and there exists a unique smooth Calabi-Yau Kähler metric $g(t) \in c_1(\alpha_t)$. The following is a natural mathematical formulation for a conjecture of Candelas and de la Ossa [32].

Conjecture 2.1. Let $T(X, Y, Y_s)$ be a conifold transition. The metric completion of (Y_{reg}, g_Y) is a compact length metric space homeomorphic to Y as a projective variety. If we denote such a metric space by (Y, d_Y) , then $(X, g(t))$ and (Y_s, g_{Y_s}) converge to (Y, d_Y) in Gromov-Hausdorff topology as $t, s \rightarrow 0$

$$(2.11) \quad (X, g(t)) \xrightarrow{d_{GH}} (Y, d_Y) \xleftarrow{d_{GH}} (Y_s, g_{Y_s}) , \quad t, s \rightarrow 0.$$

The following theorem of Rong and Zhang [32] proves a version of the above conjecture for general geometric transitions.

Theorem 2.2. Let $T(X, Y, Y_s)$ be a geometric transition. The metric completion of (Y_{reg}, g_Y) is a compact length metric space and we denote it by $(Y', d_{Y'})$. Then $(X, g(t))$ and (Y_s, g_{Y_s}) converge to $(Y', d_{Y'})$ in Gromov-Hausdorff topology as $t, s \rightarrow 0$.

The contribution of the paper is to give uniform estimates near the exceptional rational curves in the case of conifold transitions and prove the metric completion $(Y', d_{Y'})$ is homeomorphic to the projective variety Y itself (cf. Theorem 1.1).

The algebraic structure of conifold transitions are rather well understood (cf. [33]). Let $T(X, Y, Y_s)$ be a conifold transition. Y is then a normal Calabi-Yau threefold with isolated conifold singularities y_1, y_2, \dots, y_d . If $f : X \rightarrow Y$ is a minimal resolution of Y at y_1, y_2, \dots, y_d , each component of the exceptional locus $D_j = f^{-1}(y_j)$ is a smooth rational curve \mathbb{P}^1 with normal bundle

$$N_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

We define $X^\circ = X \setminus (D_1 \cup \dots \cup D_d)$ and obviously X° is isomorphic to Y_{reg} . We will try to understand the local structure near these exceptional rational curves analytically in the next section.

3. Local ansatz

In this section, we will apply the Calabi ansatz introduced by Calabi [3] (also see [27, 42]) to understand the small contraction near the exceptional locus.

Calabi ansatz. Let $E = \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$ be the holomorphic bundle over \mathbb{P}^n of rank $n+1$. Let $z = (z_1, z_2, \dots, z_n)$ be a fixed set of inhomogeneous coordinates for \mathbb{P}^n . and

$$\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log(1 + |z|^2) \in \mathcal{O}_{\mathbb{P}^n}(1)$$

be the Fubini-Study metric on \mathbb{P}^n and h be the hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$ such that $Ric(h) = -\omega_{FS}$. This induces a hermitian metric h_E on E is given by

$$h_E = h^{\oplus(n+1)}.$$

Under a local trivialization of E , we write

$$e^\rho = h_\xi(z) |\xi|^2, \quad \xi = (\xi_1, \xi_2, \dots, \xi_{n+1}),$$

where $h_\xi(z)$ is a local representation for h_E with $h_\xi(z) = (1 + |z|^2)$.

Now we are going to define a family of Kähler metrics on E as below

$$(3.12) \quad \omega = a\omega_{FS} + \sqrt{-1} \partial \bar{\partial} u(\rho)$$

for an appropriate choice of convex smooth function $u = u(\rho)$. In fact, we have the following criterion due to Calabi [3] for the above form ω to be Kähler.

Proposition 3.1. *ω defined as above, extends to a global Kähler form on E if and only if*

- (a) $a > 0$,
- (b) $u' > 0$ and $u'' > 0$ for $\rho \in (-\infty, \infty)$,
- (c) $U_0(e^\rho) = u(\rho)$ is smooth on $(-\infty, 0]$ with $U'_0(0) > 0$.

Straightforward calculations show that

$$(3.13) \quad \omega = (a + u'(\rho))\omega_{FS} + h_\xi e^{-\rho} (u' \delta_{\alpha\beta} + h_\xi e^{-\rho} (u'' - u') \xi^{\bar{\alpha}} \xi^\beta) \nabla \xi^\alpha \wedge \nabla \xi^{\bar{\beta}}.$$

Here,

$$\nabla \xi^\alpha = d\xi^\alpha + h_\xi^{-1} \partial h_\xi \xi^\alpha$$

and $\{dz^i, \nabla \xi^\alpha\}_{i=1, \dots, n, \alpha=1, 2, \dots, n+1}$ is dual to the basis

$$\nabla_{z^i} = \frac{\partial}{\partial z^i} - h_\xi^{-1} \frac{\partial h_\xi}{\partial z^i} \sum_\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha}, \quad \frac{\partial}{\partial \xi^\alpha}.$$

Let $L = \mathcal{O}_{\mathbb{P}^n}(-1)$. Then $E = L^{\oplus(n+1)}$. Let $p_\alpha : E \rightarrow L$ be the projection from E to its α th component, and let $e^{\rho\alpha} = (1 + |z|^2)|\xi_\alpha|^2$ and so $e^\rho = \sum_{\alpha=1}^{n+1} e^{\rho\alpha}$.

A local conifold flop. From now on, we assume that $n = 1$ and so E is a rank two bundle over \mathbb{P}^1 . Let P_0 be the zero section of E which is a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then by contracting P_0 , one obtains the variety \hat{E} with only one isolated double point as singularity. We can now define a flop for E by letting $E' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. \hat{E} is isomorphic to E with a similar local trivialization (w, η_1, η_2) . Then the flop between E and E'

$$(3.14) \quad \begin{array}{ccc} E & \xrightarrow{\check{f}} & E' \\ & \searrow f \quad \swarrow f' & \\ & \hat{E} & \end{array},$$

can be viewed as change of coordinates as below

$$z = \frac{\eta_2}{\eta_1}, \quad \xi_1 = \eta_1, \quad \xi_2 = w\eta_1,$$

or

$$w = \frac{\xi_2}{\xi_1}, \quad \eta_1 = \xi_1, \quad \eta_2 = z\xi_1.$$

We also have the following relation

$$e^\rho = (1 + |z|^2)(|\xi_1|^2 + |\xi_2|^2) = (1 + |w|^2)(|\eta_1|^2 + |\eta_2|^2).$$

Let $\pi : E \rightarrow \mathbb{P}^1$ and $\pi' : E' \rightarrow \mathbb{P}^1$. Then for each fixed $w \in \mathbb{P}^1$ the proper transformation of $(\pi')^{-1}(w)$ via \check{f}^{-1} is the hypersurface of E given by

$$\xi_2 = w\xi_1.$$

Such a hypersurface is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$ or simply \mathbb{C}^2 blow-up at one point and we denote it by L_w . Hence we obtained a meromorphic family of isomorphic surfaces L_w in E parametrized by $w \in \mathbb{P}^1 = \pi'(E')$. We can also view E as a meromorphic fibration of \mathbb{C}^2 blow-up at one point over \mathbb{P}^1 . The following lemma can be obtained by explicit calculations.

Lemma 3.1. *For each $w \in \mathbb{P}^1$, L_w is isomorphic to \mathbb{C}^2 blow-up at one point. Furthermore, for $w_1 \neq w_2$,*

$$L_{w_1} \cap L_{w_2} = P_0,$$

where P_0 is the zero section of E .

Local forms. We will define two reference forms on E . We first fix a Kähler form $\hat{\omega}$ on E by

$$\begin{aligned} \hat{\omega} &= \omega_{FS} + \sqrt{-1}\partial\bar{\partial}e^\rho \\ &= (1 + (1 + |z|^2)e^\rho)\omega_{FS} + \sqrt{-1}z\bar{\xi}d\xi \wedge d\bar{z} + \sqrt{-1}\bar{z}\xi dz \wedge d\bar{\xi} + \sqrt{-1}(1 + |z|^2)d\xi \wedge d\bar{\xi}. \end{aligned}$$

Then we choose a smooth closed nonnegative real $(1, 1)$ form τ defined by

$$\begin{aligned}\tau &= \sqrt{-1}\partial\bar{\partial}e^{\rho_1} \\ &= \sqrt{-1}|\xi_1|^2 dz \wedge d\bar{z} + \sqrt{-1}z\bar{\xi}_1 d\xi_1 \wedge d\bar{z} + \sqrt{-1}\bar{z}\xi_1 dz \wedge d\bar{\xi}_1 + \sqrt{-1}(1+|z|^2)d\xi_1 \wedge d\bar{\xi}_1 \\ &= \sqrt{-1}(1+|z|^2)e^{\rho_1}\omega_{FS} + \sqrt{-1}z\bar{\xi}_1 d\xi_1 \wedge d\bar{z} + \sqrt{-1}\bar{z}\xi_1 dz \wedge d\bar{\xi}_1 + \sqrt{-1}(1+|z|^2)d\xi_1 \wedge d\bar{\xi}_1.\end{aligned}$$

Although τ is not big, it defines a flat degenerate Kähler form on L_w for each w .

Lemma 3.2. *Let $\nu_1 = z\xi_1$, $\nu_2 = \xi_1$, and $\nu = (\nu_1, \nu_2)$. Then $e^{\rho_1} = |\nu|^2$ and*

$$\tau = \sqrt{-1}(d\nu_1 \wedge d\bar{\nu}_1 + d\nu_2 \wedge d\bar{\nu}_2)$$

is the pullback of the flat Euclidean metric on \mathbb{C}^2 . Hence τ is flat on $L_w \setminus P_0$ for each $w \in \mathbb{P}^1$.

The restriction of $\hat{\omega}$ on L_w for fixed w is given by

$$\begin{aligned}\hat{\omega} &= \omega_{FS} + \sqrt{-1}\partial\bar{\partial}e^{\rho} = \omega_{FS} + (1+|w|^2)\sqrt{-1}\partial\bar{\partial}e^{\rho_1} \\ &= (1 + (1+|w|^2)(1+|z|^2)e^{\rho_1})\omega_{FS} \\ &\quad + \sqrt{-1}(1+|w|^2)(z\bar{\xi}_1 d\xi_1 \wedge d\bar{z} + \bar{z}\xi_1 dz \wedge d\bar{\xi}_1 + (1+|z|^2)d\xi_1 \wedge d\bar{\xi}_1).\end{aligned}$$

We are only interested the local behavior of these forms near the zero section P_0 , so we define

$$(3.15) \quad \Omega = \{\rho < 0\} \subset E,$$

or equivalently in local coordinates,

$$e^{\rho} = (1+|z|^2)|\xi|^2 = |\nu|^2 + (1+|\nu_1/\nu_2|^2)|\xi_2|^2 \leq 1.$$

Then

$$(3.16) \quad L_w \cap \Omega = \{(z, \xi_1, \xi_2) \mid \xi_2 = w\xi_1, e^{\rho_1} \leq (1+|w|^2)^{-1}\}.$$

We now compare $\hat{\omega}$ and τ on each meromorphic fibre $L_w \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$.

Lemma 3.3. *For all $w \in \mathbb{C}$,*

$$(3.17) \quad \tau|_{L_w \cap \Omega} \leq \hat{\omega}|_{L_w \cap \Omega} \leq 2e^{-\rho_1}\tau|_{L_w \cap \Omega}.$$

Proof. The lower bound for $\hat{\omega}$ is trivial because

$$\hat{\omega} \geq \sqrt{-1}\partial\bar{\partial}e^{\rho} \geq \sqrt{-1}\partial\bar{\partial}e^{\rho_1} = \tau.$$

Restricted on each $L_w \cap \Omega$,

$$\sqrt{-1}\partial\bar{\partial}e^{\rho} = (1+|w|^2)\sqrt{-1}\partial\bar{\partial}e^{\rho_1} \leq e^{-\rho_1}\tau,$$

because $1+|w|^2 \leq e^{-\rho_1}$ by (3.16). We also have on L_w ,

$$e^{\rho_1}\omega_{FS} \leq \frac{\sqrt{-1}|\nu_2|^2}{1+|\nu_1/\nu_2|^2}d\left(\frac{\nu_1}{\nu_2}\right) \wedge d\left(\overline{\frac{\nu_1}{\nu_2}}\right) \leq \sqrt{-1}\partial\bar{\partial}|\nu|^2 = \tau.$$

The lemma follows immediately as

$$\hat{\omega} = \omega_{FS} + \sqrt{-1}\partial\bar{\partial}e^{\rho} \leq 2e^{-\rho_1}\tau$$

restricted on $L_w \cap \Omega$.

□

We also remark that the estimate (3.17) still holds if one changes the trivialization by $U(2)$ action on $\xi = (\xi_1, \xi_2)$, because $\hat{\omega}$ is invariant by $U(2)$ -action and the bounding coefficients do not depend on the choice of local trivialization as long as they are equivalent by $U(2)$ -action.

A local model. We will now construct a family of complete Ricci-flat Kähler metrics on E . Such metrics are given in [5] and here we give the calculations in terms of the Calabi ansatz. Let $\omega_E(t) = t\omega_{FS} + \sqrt{-1}\partial\bar{\partial}u$ be a Kähler metric with Calabi symmetry defined on E for $t \in (0, 1]$. Then the Ricci curvature of $\omega_E(t)$ is given by

$$\text{Ric}(\omega_E(t)) = -\sqrt{-1}\partial\bar{\partial}(\log(t + u')u'u'' - 2\rho).$$

The vanishing Ricci curvature is equivalent the following equation

$$((t + u')u'u'')' = e^{2\rho},$$

and then by integration twice, we have

$$(3.18) \quad 2(u')^3 + 3t(u')^2 - 3e^{2\rho} = 0.$$

For each $t > 0$, equation (3.18) can be explicitly solved for u' by the cubic formula and it is asymptotically of order $t^{-1/2}e^\rho$ near $\rho = -\infty$.

when $t = 0$, equation (3.18) becomes $(u')^3 = 3e^{2\rho}/2$ and the solution is explicitly given by

$$u'_{\hat{E}} = (3/2)^{1/3}e^{2\rho/3}, \quad u''_{\hat{E}} = (2/3)^{2/3}e^{2\rho/3}.$$

Such $u_{\hat{E}}$ induces a complete Ricci-flat Kähler metric

$$(3.19) \quad \omega_{CY, \hat{E}} = \sqrt{-1}\partial\bar{\partial}u_{\hat{E}}$$

on \hat{E} with an isolated cone singularity.

4. Estimates

From now on, we consider the small contraction morphism

$$\pi : X \rightarrow Y$$

from a smooth Calabi-Yau threefold X to a conifold Y . Without loss of generality, we assume that y_1, \dots, y_d are all the ordinary double points of Y with $D_i = \pi^{-1}(y_j)$ for $j = 1, \dots, d$.

Due to the estimates in Proposition 2.1 away from the exceptional rational curves D_1, \dots, D_d , it suffices to prove a uniform estimate for the degenerating family of Calabi-Yau metrics in a small neighborhood of each exceptional rational curve. Without loss of generality, we localize the problem by looking at a neighborhood of a fixed irreducible rational curve D isomorphic to

$$\Omega = E \cap \{\rho < 0\}$$

with $D = \{\rho = -\infty\}$ as defined in (3.15).

Let $\omega(t)$ be the Ricci-flat Kähler metric on X defined in (2.4) for $t \in (0, 1]$ with the same assumptions. We will restrict all the metrics and apply estimates to Ω . Then for each $t \in (0, 1]$, $\omega(t)$ is equivalent to $\hat{\omega}$ on Ω .

The goal in the section is to obtain a second order estimate for the local potential of $\omega(t)$. The usual method in [49, 40] does not quite work in this case as there does not exist a good reference metric with admissible curvature properties, in particular, some component in the curvature tensor of $\hat{\omega}$ tends $-\infty$ near P_0 . The geometric interpretation of such difficulty is that the degenerate locus for the complex Monge-Ampère equation (2.4) has codimension greater than one. Since E admits a meromorphic family of \mathbb{C}^2 blow-up at one point as shown in section 3, we consider a partial 2nd order estimate by bounding the metric along each meromorphic fibre. We therefore take advantage of the geometric flop structure and apply the maximum principle by a meromorphic slicing, so that the exceptional locus restricted to each meromorphic fibre has codimension one and we can apply ideas in [40]. More precisely, for any Kähler form ω on E , we can take the fibre-wise trace of ω with respect to τ along each L_w .

Definition 4.1. We define for $t \in (0, 1]$,

$$H(t, \cdot) = \text{tr}_{\tau|_{L_w \cap \Omega}}(\omega(t)|_{L_w \cap \Omega}).$$

Here τ and $\omega(t)$ are restricted to L_w as smooth real closed $(1, 1)$ -forms. H can also be expressed as

$$H(t, \cdot) = \frac{\omega(t) \wedge \tau \wedge dw \wedge d\bar{w}}{\tau^2 \wedge dw \wedge d\bar{w}}.$$

Lemma 4.1. $H \in C^\infty(\bar{\Omega} \setminus S)$ for all $t \in (0, 1]$, where $S = \{\rho_1 = -\infty\}$. Furthermore,

(a) for all $t \in (0, 1]$,

$$\sup_{\Omega} e^{\rho_1} H(t, \cdot) < \infty;$$

(b) there exists $C > 0$ such that for all $t \in (0, 1]$,

$$\sup_{\partial\Omega} e^{\rho_1} H(t, \cdot) \leq C.$$

Proof. For each fixed $t \in (0, 1]$, $\omega(t)$ is equivalent to $\hat{\omega}$ on Ω and so (a) follows immediately from Lemma 3.3. By Proposition 2.1, there exists $C > 0$ such that for all $t \in (0, 1]$ and $p \in \partial\Omega$,

$$\omega(t) \leq C\hat{\omega}.$$

Therefore for all $t \in (0, 1]$,

$$\sup_{\partial\Omega} e^{\rho_1} H(t, \cdot) \leq C \sup_{\partial\Omega} e^{\rho_1} \text{tr}_{\tau|_{L_w \cap \Omega}}(\hat{\omega}|_{L_w \cap \Omega}) \leq 2C$$

and it proves (b). □

Proposition 4.1. Let Δ_t be the Laplace operator associated to the Ricci-flat Kähler metric $g(t)$ for $t \in (0, 1]$. Then

$$(4.20) \quad \Delta_t \log H \geq 0.$$

Proof. We define

$$I = \log H$$

and break the proof into the following steps.

Step 1. We first make a choice of special coordinates. On Ω , we have the standard local coordinates with Calabi symmetry as defined in the previous section, i.e., for each $p \in \Omega$, we have at p , $(z(p), \xi_1(p), \xi_2(p))$. Once we fix p , there exists a unique $w \in \mathbb{P}^1$ such that $p \in L_w$.

- (a) Near $p \in \Omega$, we first choose the coordinates (ν_1, ν_2, w) , where $\nu_1 = \xi_1$ and $\nu_2 = z\xi_1$ as defined in Lemma 3.2. We will apply a linear transformation to (ν_1, ν_2, w) such that

$$x = (x_1, x_2, x_3)^T = A^{-1}(\nu_1, \nu_2, w)^T.$$

We assume that A is in the form of

$$\begin{pmatrix} A' & a \\ 0 & 1 \end{pmatrix},$$

where A' is a 2×2 matrix and a is a 2×1 vector. Immediately, we have

$$x_3 = w.$$

- (b) Suppose $g(t)$ at (t, p) is given by the following hermitian matrix with respect to coordinates (ν_1, ν_2, w)

$$G = \begin{pmatrix} B & b \\ \bar{b}^T & c \end{pmatrix},$$

where B is a 2×2 hermitian matrix, b a 2×1 vector. Then under the new coordinates x , $g(t)$ at p is given by the following hermitian matrix

$$\begin{aligned} \bar{A}^T G A &= \begin{pmatrix} \bar{A}'^T B A' & \bar{A}'^T B a + \bar{A}'^T b \\ \bar{a}^T B A' + \bar{b}^T A' & \bar{a}^T B a + \bar{a}^T b + \bar{b}^T a + c \end{pmatrix} \\ &= \begin{pmatrix} \bar{A}'^T B A' & \bar{A}'^T (B a + b) \\ (\bar{a}^T B + \bar{b}^T) A' & \bar{a}^T B a + 2\text{Re}(\bar{a}^T b) + c \end{pmatrix}. \end{aligned}$$

- (c) We choose a unitary matrix A' such that $\bar{A}'^T B A'$ is diagonalized, i.e.,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and choose a such that

$$B a = -b$$

since B has rank 2. Therefore under the coordinates x , at (t, p) ,

$$g = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda_3 = c - \bar{a}^T B a$. The matrix representation of τ under the coordinates (ν_1, ν_2, w) is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so its matrix representation under the coordinates X at (t, p) is given by

$$\tau = \bar{A}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} I_{2 \times 2} & \bar{A}^T a \\ \bar{a}^T A' & \bar{a}^T a \end{pmatrix}$$

since A' is unitary. Since $x_3 = w$ and on L_w , x_3 is constant and $\omega|_{L_w \cap \Omega} = \sqrt{-1} \sum_{i,j=1}^2 g_{i\bar{j}} dx^i \wedge d\bar{x}^j$. Then at (t, p) ,

$$g|_{L_w} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \tau|_{L_w} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, we arrive at

$$I(t, p) = \log \sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} (g|_{L_w})_{i\bar{j}} = \log(\lambda_1 + \lambda_2).$$

Step 2. Now we calculate $\Delta_t I$ at (t, p) under the coordinates x . Notice that $\tau|_{L_w}$ is a constant form $\sqrt{-1}(dx_1 \wedge d\bar{x}_1 + dx_2 \wedge d\bar{x}_2)$ and so all derivatives of $\tau|_{L_w}$ vanish.

We now apply the Laplace operator Δ_t to H .

$$\begin{aligned} & \Delta_t H \\ &= \sum_{k,l=1}^3 g^{k\bar{l}} \left(\sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j}} \right)_{k\bar{l}} \\ &= \sum_{k,l=1}^3 \sum_{i,j=1,2} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j},k\bar{l}} - \sum_{k,l=1}^3 \sum_{i,j,p,q=1,2} g^{k\bar{l}} g_{i\bar{j}} (\tau|_{L_w})^{i\bar{q}} (\tau|_{L_w})^{p\bar{j}} \tau_{p\bar{q},k\bar{l}} \\ &= - \sum_{k,l=1}^3 \sum_{i,j=1,2} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} R_{i\bar{j},k\bar{l}} + \sum_{k,l,p,q=1}^3 \sum_{i,j=1,2} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{i\bar{q},k} \\ &= - \sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} R_{i\bar{j}} + \sum_{k,l,p,q=1}^3 \sum_{i,j=1,2} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{i\bar{q},k} \\ &= \sum_{k,l,p,q=1}^3 \sum_{i,j=1,2} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{l},\bar{j}} g_{k\bar{q},i}. \end{aligned}$$

Then

$$\begin{aligned}
& \Delta_t I \\
&= (H)^{-1} \sum_{k,l=1}^3 \sum_{i,j=1,2;p,q=1,2,3} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{i\bar{q},k} - (H)^{-2} |\nabla I|_g^2 \\
&= (H)^{-2} \sum_{k,l=1}^3 \left(H \sum_{i,j=1,2;p,q=1,2,3} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{i\bar{q},k} - g^{k\bar{l}} \left(\sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j},k} \right) \left(\sum_{i,j=1,2} (\tau|_{L_w})^{j\bar{i}} g_{j\bar{i},\bar{l}} \right) \right) \\
&= (H)^{-2} \sum_{k,l=1}^3 \left(H \sum_{i,j=1,2;p,q=1,2,3} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{k\bar{q},i} - g^{k\bar{l}} \left(\sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j},k} \right) \left(\sum_{i,j=1,2} (\tau|_{L_w})^{j\bar{i}} g_{j\bar{i},\bar{l}} \right) \right).
\end{aligned}$$

Step 3. The proof of the proposition is now reduced to show that

$$\sum_{k,l=1}^3 \left(H \sum_{i,j=1,2;p,q=1,2,3} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{k\bar{q},i} - g^{k\bar{l}} \left(\sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j},k} \right) \left(\sum_{i,j=1,2} (\tau|_{L_w})^{j\bar{i}} g_{j\bar{i},\bar{l}} \right) \right) \geq 0.$$

Note that $\tau_{i\bar{j}} = \delta_{ij}$ for $i, j = 1, 2$ and $g = (\lambda_1, \lambda_2, \lambda_3)$. Then

$$\begin{aligned}
& \sum_{k,l=1}^3 g^{k\bar{l}} \left(\sum_{i,j=1,2} (\tau|_{L_w})^{i\bar{j}} g_{i\bar{j},k} \right) \left(\sum_{i,j=1,2} (\tau|_{L_w})^{j\bar{i}} g_{j\bar{i},\bar{l}} \right) \\
&= \sum_{k=1,2,3} \lambda_k^{-1} \left| \sum_{i=1,2} g_{i\bar{i},k} \right|^2 \\
&\leq \sum_{i,j=1,2} \left(\sum_{k=1,2,3} \lambda_k^{-1} |g_{i\bar{i},k}|^2 \right)^{1/2} \left(\sum_{k=1,2,3} \lambda_k^{-1} |g_{j\bar{j},k}|^2 \right)^{1/2} \\
&= \left(\sum_{i=1,2} \left(\sum_{k=1,2,3} \lambda_k^{-1} |g_{i\bar{i},k}|^2 \right)^{1/2} \right)^2 \\
&= \left(\sum_{i=1,2} \lambda_i^{1/2} \left(\sum_{k=1,2,3} \lambda_k^{-1} \lambda_i^{-1} |g_{i\bar{i},k}|^2 \right)^{1/2} \right)^2 \\
&\leq \left(\sum_{i=1,2} \lambda_i \right) \left(\sum_{i=1,2;k=1,2,3} \lambda_k^{-1} \lambda_i^{-1} |g_{i\bar{i},k}|^2 \right) \\
&\leq H \left(\sum_{k,l=1,2,3;i=1,2} \lambda_k^{-1} \lambda_l^{-1} |g_{i\bar{l},k}|^2 \right) \\
&= H \sum_{i,j=1,2;k,l,p,q=1,2,3} g^{k\bar{l}} (\tau|_{L_w})^{i\bar{j}} g^{p\bar{q}} g_{p\bar{j},\bar{l}} g_{i\bar{q},k}.
\end{aligned}$$

This completes the proof of the proposition.

□

Corollary 4.1. *There exists $C > 0$ such that on Ω , for all $t \in (0, 1]$,*

$$(4.21) \quad H \leq C e^{-\rho_1}.$$

Proof. Let

$$I_\epsilon = \log H + (1 + \epsilon)\rho_1$$

for $\epsilon > 0$. Let $S = \{\rho_1 = -\infty\}$. Then for all $\epsilon > 0$, $\limsup_{p \rightarrow S} I_\epsilon = -\infty$ by Lemma 4.1, and on $\Omega \setminus S$,

$$\Delta_t I_\epsilon > 0,$$

because of Proposition 4.1 and the fact that $\Delta_t \rho_1 = \Delta_t \log(1 + |z|^2) |\xi_1|^2 = \text{tr}_\omega(\omega_{FS}) > 0$ on $\Omega \setminus S$.

Applying the maximum principle for I_ϵ , we know that the maximum of I_ϵ has to be achieved on $\partial\Omega$. Then by Lemma 4.1, there exists $C > 0$ such that for all $\epsilon \in (0, 1]$ and $t \in (0, 1]$,

$$\sup_{\Omega \setminus S} I_\epsilon = \sup_{\partial\Omega} I_\epsilon \leq \sup_{\partial\Omega} I_0 \leq C.$$

The corollary is then proved by letting $\epsilon \rightarrow 0$.

□

We define a holomorphic vector V on Ω by

$$V = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}.$$

V vanishes along P_0 and

$$|V|_{\hat{\omega}}^2 = e^\rho.$$

We also consider the normalized vector field

$$W = \frac{V}{|V|_{\hat{\omega}}} = e^{-\rho/2} \sum_{\alpha=1,2} \xi_\alpha \frac{\partial}{\partial \xi_\alpha}.$$

Now we can obtain uniform bounds for the degenerating Ricci-flat Kähler metrics $g(t)$ near the exceptional curves.

Proposition 4.2. *There exists $C > 0$ such that for all $t \in (0, 1]$ and on Ω ,*

$$(4.22) \quad C^{-1} \omega_{\hat{E}} \leq \omega(t) \leq C e^{-\rho} \omega_{\hat{E}},$$

and

$$(4.23) \quad |W|_{g(t)}^2 \leq C e^{-\rho/2},$$

where $\omega_{\hat{E}} = \sqrt{-1} \partial \bar{\partial} e^\rho$.

Proof. We break the proof into the following steps.

Step 1. We apply the similar argument in the proof of Schwarz lemma [50, 36]. Notice that $\omega(t) = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi$ with $\varphi \in C^\infty(X)$ uniformly bounded in $L^\infty(X)$ for $t \in (0, 1]$. Also there exists $C_1 > 0$ such that for all $t \in (0, 1]$ and on Ω , $\omega_t \geq C_1 \omega_{\hat{E}}$ on Ω . Then we consider the quantity

$$L = \log \text{tr}_\omega(\omega_{\hat{E}}) - \varphi.$$

$\omega_{\hat{E}}$ restricted to Ω is in fact the pullback of a flat metric on \mathbb{C}^4 given by a local morphism $(\xi_1, \xi_2, z\xi_1, z\xi_2)$. Then straightforward calculations give

$$\Delta_t L \geq tr_\omega(\omega_t) - 3 \geq tr_\omega(\omega_{\hat{E}}) - 3.$$

Applying the maximum principle, we have

$$tr_\omega(\omega_{\hat{E}}) \leq \sup_{\partial\Omega} tr_\omega(\omega_{\hat{E}}) + 3.$$

Note that $tr_{\omega(t)}(\omega_{\hat{E}})$ is uniformly bounded on $\partial\Omega$. Hence $tr_\omega(\omega_{\hat{E}})$ is uniformly bounded above and so there exists $C_1 > 0$ such that

$$(4.24) \quad \omega \geq C_1 \omega_{\hat{E}}.$$

Step 2. Since ω^3 is uniformly equivalent to $\hat{\omega}^3$ and $e^{-\rho}\omega_{\hat{E}}^3$ in Ω , there exists $C_2 > 0$ such that

$$(4.25) \quad \omega^3 \leq C_2 e^{-\rho} \omega_{\hat{E}}^3.$$

By the estimates (4.24) and (4.25), there exists $C_3 > 0$ such that

$$tr_{\omega_{\hat{E}}}(\omega) \leq C_3 e^{-\rho}$$

and so $\omega \leq C_3 e^{-\rho} \omega_{\hat{E}}$. This completes the proof for estimate (4.22).

Step 3. Let $V_1 = \xi_1 \frac{\partial}{\partial \xi_1}$ be the holomorphic vector field on Ω . Then V_1 vanishes on $\rho_1 = -\infty$ and $|V_1|_{\hat{\omega}}^2 = (1 + |z|^2)|\xi_1|^2 = e^{\rho_1}$. Using the normal coordinates for ω , we can show that

$$\Delta_t |V_1|_g^2 = -(V_1)^i (V_1)^{\bar{j}} R_{i\bar{j}} + g^{k\bar{l}} g_{i\bar{j}} \left((V_1)^i \right)_k \left((V_1)^{\bar{j}} \right)^{\bar{l}} = |\partial V|_g^2$$

and so

$$\Delta_t \log |V_1|_g^2 = (|V_1|_g^2)^{-2} (|V_1|_g^2 |\partial V_1|_g^2 - |\nabla_t |V_1|_g|^2) \geq 0.$$

We now define

$$G_\epsilon = \log \left(e^{\epsilon \rho_1} |V_1|_{\omega}^2 tr_{\tau|_{L_w}}(\omega|_{L_w}) \right) = I + \log |V_1|_{\omega}^2 + \epsilon \rho_1$$

for $\epsilon \in (0, 1]$. For each $t \in (0, 1]$, G_ϵ is smooth in Ω away from $\rho_1 = -\infty$, and it tends to $-\infty$ near $\rho_1 = -\infty$ for all $\epsilon \in (0, 1]$ by Lemma 4.1. Furthermore, there exists $C_4 > 0$ such that for all $\epsilon \in (0, 1]$ and $t \in (0, 1]$,

$$\sup_{\partial\Omega} G_\epsilon \leq C_4.$$

On the other hand,

$$\Delta_t G_\epsilon = \Delta_t I + \Delta_t \log |V_1|_{\omega}^2 + \epsilon \Delta_t \rho_1 > 0.$$

By the maximum principle,

$$\sup_{\Omega} G_\epsilon \leq \sup_{\partial\Omega} G_\epsilon \leq C_4.$$

Then by letting ϵ tend to 0, we have

$$(4.26) \quad |V_1|_{\omega}^2 tr_{\tau|_{L_w}}(\omega|_{L_w}) \leq C_4.$$

Step 4. We will apply the estimate (4.26) to prove (4.23). Under the coordinates (w, ν_1, ν_2) , we have

$$(\tau|_{L_w})_{\nu_i \bar{\nu}_j} = \delta_{ij}, \quad V_1 = \nu_1 \frac{\partial}{\partial \nu_1} + \nu_2 \frac{\partial}{\partial \nu_2} - w \frac{\partial}{\partial w}, \quad |V_1|_\tau^2 = |\nu|^2 = e^{\rho_1},$$

At any $p \in L_w$ with $w = 0$, we have

$$V_1 = V, \quad \rho_1 = \rho,$$

$$\begin{aligned} |V_1|_g^2 &= |\nu_1|^2 g_{\nu_1 \bar{\nu}_1} + 2\operatorname{Re}(\nu_1 \bar{\nu}_2 g_{\nu_1 \bar{\nu}_2}) + |\nu_2|^2 g_{\nu_2 \bar{\nu}_2} \\ &\leq (|\nu_1|^2 + |\nu_2|^2) (g_{\nu_1 \bar{\nu}_1} + g_{\nu_2 \bar{\nu}_2}) \\ &= e^{\rho_1} \operatorname{tr}_{\tau|_{L_0}}(\omega|_{L_0}). \end{aligned}$$

Combined with (4.26), there exists $C_4 > 0$ such that for all $t \in (0, 1]$ and on $L_0 \cap \Omega$,

$$|V_1|_\omega^2 \leq C_4 e^{\rho_1/2}, \quad \text{or} \quad |V|_\omega^2 \leq C_4 e^{\rho/2}.$$

Equivalently, we have,

$$(|W|_\omega^2)|_{L_0 \cap \Omega} \leq C_7 e^{-\rho/2}|_{L_0 \cap \Omega}.$$

Notice that W, V, ρ are $U(2)$ -invariant in terms of ξ and all the bounds we have derived do not depend on the choice of trivialization differing by $U(2)$ -action. Therefore we have

$$|W|_\omega^2 \leq C_7 e^{-\rho/2}$$

uniformly for $t \in (0, 1]$ and Ω . This completes the proof of the proposition. \square

The uniform bound on $\operatorname{diam}(X, g(t))$ is already known to the general Calabi-Yau degeneration due to [46]. The following corollary follows from Proposition 4.2.

Corollary 4.2. *There exists $C > 0$ such that for all $t \in (0, 1]$,*

$$(4.27) \quad \operatorname{diam}(X, g(t)) \leq C, \quad \operatorname{diam}(X \setminus \{D_1, \dots, D_d\}, g(t)) \leq C.$$

Proof. Since $|W|_{g(t)}^2 \leq C e^{-\rho/2}$, any point $p = (z, \xi)$ in Ω can be connected by a radial path γ_p defined by

$$(z, s\xi), \quad s \in \left[0, \frac{1}{(1+|z|^2)|\xi|^2}\right]$$

to P_0 and $\partial\Omega$ with $p' = \gamma \left(\frac{1}{(1+|z|^2)|\xi|^2} \right) \in \partial\Omega$. Then the arc length of γ_p with respect to $g(t)$ is uniformly bounded, i.e., there exists $C' > 0$ such that for all $t \in (0, 1]$ and for all $p \in \Omega$,

$$|\gamma_p|_{g(t)} \leq C'.$$

On the other hand, $g(t)$ is uniformly equivalent to $\hat{\omega}$ on $\partial\Omega$. Given any two points $p, q \in \Omega$, we can join p, q by γ_p, γ_q and a smooth geodesic path $\gamma_{p', q'}$ with respect to $\hat{\omega}$ joining p' and q' in $\partial\Omega$. Therefore, both $\operatorname{diam}(\Omega, g(t))$ and $\operatorname{diam}(\Omega \setminus P_0, g(t))$ are uniformly bounded and this completes the proof of the corollary. \square

The following corollary shows that the restriction of $g(t)$ to the exceptional rational curve is uniformly bounded above.

Corollary 4.3. *There exists $C > 0$ such that for all $t \in (0, 1]$,*

$$(4.28) \quad \omega(t)|_{P_0} \leq C\omega_{FS}|_{P_0}.$$

Proof. By (4.22) in Proposition 4.2, there exist $C_1, C_2 > 0$ such that for all $t \in (0, 1]$ and on Ω ,

$$\frac{\omega \wedge \omega_{\hat{E}}^2}{\omega_{FS} \wedge \omega_{\hat{E}}^2} \leq C_1 e^{-\rho} \frac{\omega_{\hat{E}}^3}{\omega_{FS} \wedge \omega_{\hat{E}}^2} \leq C_2.$$

For any point $p \in P_0$, there exist $e_1 \in T_p P_0$ and $e_2, e_3 \in T_p E$ such that they form an orthonormal basis of $T_p E$ with respect to $\hat{\omega}$. Obviously,

$$\omega_{\hat{E}}(e_1, \cdot) = 0.$$

Then

$$tr_{\omega_{FS}|_{P_0}}(\omega|_{P_0}) = \frac{\omega(e_1 \wedge \bar{e}_1)}{\omega_{FS}(e_1 \wedge \bar{e}_1)} = \frac{\omega \wedge \omega_{\hat{E}}^2(e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_3 \wedge \bar{e}_3)}{\omega_{FS} \wedge \omega_{\hat{E}}^2(e_1 \wedge \bar{e}_1 \wedge \cdots \wedge e_3 \wedge \bar{e}_3)} \leq C_2.$$

□

In fact, the following proposition shows that exceptional rational curve become extinct as $t \rightarrow 0$.

Proposition 4.3. *There exists $C > 0$ such that for all $t \in (0, 1]$ such that*

$$(4.29) \quad diam(P_0, g(t)|_{P_0}) \leq Ct^{1/3}.$$

Proof. There exists $C > 0$ such that for all $t \in (0, 1]$,

$$\int_{P_0} \omega(t) = P_0 \cdot [\alpha_t] = tP_0 \cdot [\alpha] \leq Ct.$$

Then the proposition is proved by the same argument in the proof of Lemma 3.2 in [40].

□

We define for $r > 0$,

$$(4.30) \quad \Omega_r = \{(z, \xi) \in E \mid e^\rho = (1 + |z|^2)|\xi|^2 \leq r^2\}.$$

Then we have the following proposition.

Proposition 4.4. *For any $\epsilon > 0$, there exist $\varepsilon > 0$ and $\delta > 0$ such that for all $t \in (0, \varepsilon)$,*

$$(4.31) \quad diam(\Omega_\delta, g(t)) < \epsilon.$$

Proof. Given any two points $p, q \in \Omega_\delta$, there exist $p', q' \in P_0$ such that p and q can be connected to p' and q' by radial paths $\gamma_{p,p'}$ and $\gamma_{q,q'}$. For any $\epsilon > 0$, we choose $\delta > 0$ such that the arc length of $\gamma_{p,p'}$ and $\gamma_{q,q'}$ is smaller than $\epsilon/3$ with respect to $g(t)$ for all $t \in (0, 1]$ by applying (4.23) in Proposition 4.2. By Proposition 4.3, we can choose $\varepsilon > 0$ sufficiently small such that for $t \in (0, \varepsilon)$,

$$diam(P_0, g(t)|_{P_0}) \leq \frac{\epsilon}{3}.$$

Then

$$\text{dist}_{g(t)}(p, q) \leq \text{dist}_{g(t)}(p, p') + \text{dist}_{g(t)}(q, q') + \text{diam}(P_0, g(t)|_{P_0}) < \epsilon$$

and the proposition follows. \square

5. Proof of Theorem 1.1 and its generalizations

Let $T(X, Y, Y_s)$ be a conifold transition. Let g_Y be the unique singular Calabi-Yau Kähler metric associated to the Kähler current on Y as defined in section 2. Note that g_Y is smooth in $Y_{\text{reg}} = Y \setminus \{y_1, \dots, y_d\}$ and so we define a similar distance function on Y as in Definition 5.1 in [40].

Definition 5.1. We extend g_Y on Y_{reg} to a nonnegative $(1,1)$ -tensor \tilde{g}_Y on the whole space Y by setting $\tilde{g}_Y|_{y_j}(\cdot, \cdot) = 0$ for $j = 1, \dots, d$. Of course, \tilde{g}_Y may be discontinuous at y_1, \dots, y_d . Define a distance function $d_Y : Y \times Y \rightarrow \mathbb{R}$ by

$$(5.32) \quad d_Y(y, y') = \inf_{\gamma} \int_0^1 \sqrt{g_Y(\gamma'(s), \gamma'(s))} ds,$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0, 1] \rightarrow Y$ with $\gamma(0) = y$, $\gamma(1) = y'$.

The goal is to show that such a metric space is exactly the Gromov-Hausdorff limit of $(X, g(t))$ as $t \rightarrow 0$ and it is isomorphic to the metric completion of (Y_{reg}, g_Y) .

Theorem 5.1. (Y, d_Y) is a compact metric space homeomorphic to the projective variety Y itself. Furthermore, $(X, g(t))$ converges to (Y, d_Y) in Gromov-Hausdorff topology as $t \rightarrow 0$.

Proof. The same argument in section 3 in [40] can be applied to prove the proposition with uniform estimates from Proposition 4.2 and Proposition 4.3. \square

Theorem 5.2. Let $(\tilde{Y}, d_{\tilde{Y}})$ be the metric completion of (Y_{reg}, g_Y) . Then $(\tilde{Y}, d_{\tilde{Y}})$ is isomorphic to (Y, d_Y) .

Proof. There are two ways to complete the proof of the theorem. The first approach is purely analytic. We can modify the argument in [41] to show that $(\tilde{Y}, d_{\tilde{Y}})$ is homeomorphic to Y as a projective variety and indeed, $(X, g(t))$ converges to (Y, d_Y) in Gromov-Hausdorff topology. The details can also be found in [41] and in [35] for higher codimensional surgeries by the Kähler-Ricci flow. This approach does not make use of the general theory on Riemannian manifolds with bounded Ricci curvature [7, 8, 9, 10] and it gives explicit estimates to understand the analytic and geometric contractions.

The second approach relies on the results in [32]. By Theorem 2.2, $(X, g(t))$ converges to $(\tilde{Y}, d_{\tilde{Y}})$ in Gromov-Hausdorff topology, and hence by the uniqueness of the limiting metric space,

$$(Y, d_Y) = (\tilde{Y}, d_{\tilde{Y}}).$$

The theorem follows from Theorem 5.2. \square

Proof of main results. Theorem 1.1 follows immediately from Theorem 5.2 and Theorem 5.1. Corollary 1.1 is a straightforward consequence of Theorem 1.1. Corollary is proved by combining Theorem 1.1 and Theorem 2.2.

Discussions. We now discuss some generalizations of Theorem 1.1 and future questions. First of all, Theorem 1.1, Corollary 1.1 and 1.2 can be generalized to higher dimensional conifold small contractions, flops and transitions with little modification. The same estimates can be applied to high codimensional surgery by the Kähler-Ricci flow if the exceptional locus is \mathbb{P}^n with normal bundle $\mathcal{O}_{\mathbb{P}^n}(-a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(-a_{m+1})$ for $a_i \in \mathbb{Z}^+$. Consequentially, it is shown in [35] that the Kähler-Ricci flow performs certain family of flipping contractions and resolution, in Gromov-Hausdorff topology. The blow-up limit of Type I singularities of the Ricci flow is a complete shrinking Ricci soliton [22, 12]. In the case of the Kähler-Ricci flow, finite time singularity arises from contraction of special rational curves. It is natural to conjecture that the curvature tensor blows up at the same rate as the extinction rate of such rational curves. We now make the following conjecture.

Conjecture 5.1. *With the same assumptions in Theorem 1.1, there exists $C > 0$ such that for all $t \in (0, 1]$, such that the curvature tensor $Rm(t)$ of $g(t)$ is bounded as below*

$$(5.33) \quad \sup_X |Rm(t)|_{g(t)} \leq Ct^{-1}.$$

Furthermore, the rescaled Ricci-flat Kähler metrics $\tilde{g}(t)$ of $g(t)$ converge to the Ricci-flat Kähler metric $g_{CY, \hat{E}}$ on \hat{E} with an isolated cone singularity given by (3.19), in pointed Gromov-Hausdorff topology.

In particular, the metric singularity of g_Y near the ordinary double point should be asymptotically close to the local model given by (3.19).

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